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**OUTLINE OF RESEARCH IN LIFE  
AND FATIGUE TESTING**

**DEPARTMENT OF MATHEMATICS  
WAYNE UNIVERSITY  
DETROIT, MICHIGAN**

**BENJAMIN EPSTEIN  
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# OUTLINE OF RESEARCH IN STATISTICS OF LIFE AND FATIGUE TESTING

by

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## I. Introduction

On April 1, 1951, a research project on the statistics of life and fatigue testing was initiated at Wayne University. The principal objective of the research was to develop statistical techniques of estimation and decision making which make maximum use of the information arising from life and fatigue tests. A characteristic feature of such tests is that information becomes available continuously and failures when they do occur are naturally ordered (e.g., in time). In the course of this research we have tried to develop statistical procedures in life testing which are optimum in that we get the most information out of the data for the least overall cost in time and number of items destroyed.

The research done under this project represents substantial progress in a field about which little was known two or three years ago. On the theoretical side fundamental work has been done in estimation and hypothesis testing, in stochastic processes, continuous and ordinary sequential analysis, non-parametric theory, and order statistics. The research has stimulated interest in life testing and has drawn the attention of other mathematical statisticians to some of the important problems in this field. On the applied side it has been clear from the start that the results would be use-

ful in fields as diverse as life testing of electron tubes to follow-up studies on the effectiveness of medical treatment. The potential applications are very numerous indeed.

In the sequel we shall give the following:

- Section II: Personnel on the project.
- Section III: Reports and papers resulting from the project.
- Section IV: Summary of the results obtained under the project.
- Section V: Related work by other mathematical statisticians.
- Section VI: Some remarks on the applications of this work.

## II. Personnel on the project

The following members of the Wayne University staff have been connected with the research project.

Professor Benjamin Epstein, Project Director, April 1, 1951-January 31, 1954.

Professor Milton Sobel, April 1, 1951-February 1, 1952.

Mr. C. H. Kraft, September 15, 1951-June 15, 1952.

Professor C. K. Tsao, February 1, 1952-September 1, 1953.

In addition there were

Mrs. Dorothy Wolfe, Graduate Assistant, September 15, 1951-June 1, 1953,

and several student assistants who performed calculations, typed reports, did clerical work, etc.

## III. Reports and papers resulting from the project

A. The following reports have been issued:

- (1) "Some Tests Based on the First  $r$  Ordered Observations Drawn from an Exponential Distribution", by B. Epstein and M. Sobel. Issued March 1, 1952, as Wayne University Report No. 1 and Stanford University Report No. 6.
- (2) "Estimates of Mean Life Based on the  $r$ 'th Smallest Value in a Sample of Size  $n$  Drawn from an Exponential Distribution", by B. Epstein. Issued July 1, 1952, as Technical Report No. 2.
- (3) "Some Two Sample Tests Based on Ordered Observations from the Exponential Distribution", by B. Epstein and C. K. Tsao. Issued December 15, 1952, as Technical Report No. 3.
- (4) "Some Results on Truncated Life Tests in the Exponential Case", by B. Epstein. Issued February 15, 1953, as Technical Report No. 4.

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- (5) "Operating Characteristic Curves and Other Features of Truncated Life Tests with Replacement", by B. Epstein. Issued May 15, 1953, as Technical Report No. 5.
  - (6) "Tables for the Distribution of the Number of Exceedances", by B. Epstein. Issued July 15, 1953, as Technical Report No. 6.
  - (7) "An Extension of Massey's Distribution of the Maximum Deviation Between Two Sample Cumulative Step Functions", by C. K. Tsao. Issued July 15, 1953, as Technical Report No. 7.
  - (8) "Sequential Life Tests in the Exponential Case", by B. Epstein and M. Sobel. Issued January 31, 1954, as Technical Report No. 8.

B. Papers published, accepted for publication, or submitted for publication.

- (1) B. Epstein, "Statistical Problems in Life Testing", Proceedings of the Seventh Annual Convention of the American Society for Quality Control, pp. 385-398, 1953.
- (2) B. Epstein and M. Sobel, "Life Testing", Journal of the American Statistical Association 48, 486-502, 1953.
- (3) B. Epstein and C. K. Tsao, "Some Two Sample Tests Based on Ordered Observations from an Exponential Distribution", Annals of Mathematical Statistics 24, 458-466, 1953.
- (4) B. Epstein and M. Sobel, "Some Theorems Relevant to Life Testing from an Exponential Population". Accepted for publication in the Annals of Mathematical Statistics.
- (5) B. Epstein, "Some Results on Truncated Life Tests in the Exponential Case". Submitted for publication in the Annals of Mathematical Statistics.

- (6) F. Epstein, "Tables for the Distribution of the Number of Exceedances". Submitted for publication in the Annals of Mathematical Statistics.
- (7) C. K. Tsao, "An Extension of Massey's Distribution of the Maximum Deviation Between Two Sample Cumulative Step Functions". Submitted for publication in the Annals of Mathematical Statistics.

Other papers based on this research or its extensions are in various stages of preparation. Appropriate acknowledgement will be made of support by the Office of Naval Research.

C. Several papers have been presented either by title or in person at scientific meetings. These are:

- (1) "Life Testing", by B. Epstein and M. Sobel, Sampling Inspection Conference, Stanford University, August, 1951.
- (2) "Some Tests Based on the First  $r$  Ordered Observations Drawn from an Exponential Distribution", by B. Epstein and M. Sobel, Christmas meeting of the Institute of Mathematical Statistics, Boston, Massachusetts, December, 1951.
- (3) "Some Theorems Relevant to Life Testing", by B. Epstein and M. Sobel, December, 1951.
- (4) "A Method for Reducing the Time to Complete Certain Fatigue Experiments", by L. G. Johnson, December, 1951.
- (5) "Efficiency of Estimators of the Mean of an Exponential Distribution Based Only on the  $r$ 'th Smallest Observation in an Ordered Sample", by B. Epstein, Summer meeting of the Institute of Mathematical Statistics, East Lansing, Michigan, September, 1952.

- (6) "On Truncated Rules of Action", by B. Epstein, September, 1952.
- (7) "An Extension of Massey's Distribution of the Maximum Deviation Between Two Sample Cumulative Step Functions", by C. K. Tsao, September, 1952.
- (8) "Some Two Sample Tests on the Exponential Distribution", by B. Epstein and C. K. Tsao, September, 1952.
- (9) "A Non-Parametric Two Sample Life Test", by B. Epstein, Christmas meeting of the Institute of Mathematical Statistics, Chicago, Illinois, December, 1952.
- (10) "Statistical Problems in Life Testing", by B. Epstein. Invited address on session devoted to Life Testing of Electrical and Electronic Equipment at the Seventh Annual Convention of the American Society for Quality Control, Philadelphia, Pennsylvania, May, 1953. This talk was also given by invitation at a seminar on Performance and Reliability of Complex Assemblies held at Storrs, Connecticut, August 26-28, 1953.
- (11) "Sequential Life Tests in the Exponential Case", by B. Epstein and M. Sobel, Christmas meeting of the Institute of Mathematical Statistics, Washington, D.C., December, 1953.

#### IV. Summary of the results obtained under the project.

##### 1. Tests of hypotheses

As mentioned at the beginning, the principal objective of the research was to develop statistical techniques of estimation and decision making which make maximum use of the information arising from life and fatigue tests. We first sketch briefly what has been done along the lines of decision making. Unless otherwise specified, we assume in the sequel that the underlying distribution of life is given by the exponential p.d.f.

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0, \theta > 0.$$

Suppose we are interested in the problem of testing the hypothesis  $H_0: \theta = \theta_0$  against the alternative  $H_1: \theta = \theta_1 < \theta_0$ , with prescribed Type I error =  $\alpha$  and Type II error =  $\beta$ . We have given non-sequential, truncated, and purely sequential test procedures for carrying out this decision problem. In all tests we start at time  $t = t_0 = 0$  with  $n$  items drawn at random from the underlying p.d.f. If items which fail on test are not replaced, we say that we have a non-replacement situation. If items which fail on test are replaced at once by new items drawn from the same underlying p.d.f., we say that we have a replacement situation. By a non-sequential procedure we mean one where we do not make a decision, except at the very end.

Truncation of a life test may be with respect to the maximum number of failures allowed to occur before termination of the test, or with respect to a maximum allowed time of experimentation or both. As the name implies, sequential tests make use of all of the information as it becomes available. A novel and very interesting feature of sequential life test procedures is that decisions can now be made continuously in time. Among the explicit results obtained are such useful quantities as  $L(\theta)$ , the probability of accepting  $\theta = \theta_0$  when  $\theta$  is true;  $E_\theta(r)$ , the expected number of items failed



in reaching a decision; and  $E_0(t)$ , the expected waiting time to reach a decision. These results make it possible to design life test procedures for testing  $\theta_0$  against  $\theta_1$  with prescribed errors  $\alpha$  and  $\beta$  at the smallest overall cost in time and number of items failed.

All of the results are in A1, A4, A8, B1, B2, and B5. In addition, useful two sample life test procedures of a non-sequential nature are given in A3, B3. The tests are simple to apply since they can be reduced to equivalent F tests in most cases. A sequential two sample result has also been obtained and the manuscript is being prepared.

## 2. Estimation procedures

When one turns to estimation, we have obtained results in A1, B2, in the case where  $n$  items are placed on life test and where the test is continued until the  $r$ 'th failure occurs. Let  $x_{i,n}$  be the total time (measured from the beginning of the experiment) which elapses until the  $i$ 'th failure occurs. Then it can be proved in the non-replacement case that

$$(1) \quad \hat{\theta}_{r,n} = \frac{x_{1,n} + x_{2,n} + \dots + x_{r,n} + (n-r)x_{r,n}}{r}$$

is a "best" estimate of  $\theta$  in the sense of being maximum likelihood, minimum variance, efficient, and sufficient. In the replacement case the "best" estimate is

$$(2) \quad \hat{\theta}_{r,n} = n x_{r,n} / r.$$

In either case  $2r \hat{\theta}_{r,n} / \theta$  is distributed as  $\chi^2(2r)$ . From the latter fact we have a ready means for finding confidence interval estimates for  $\theta$  based on knowledge of the first  $r$  failure times.

In the previous paragraph the life test was continued until a preassigned number of failures occur. Suppose experimentation is initiated with  $n$  items and truncated at a preassigned time  $t^*$ . In this case,  $k$ , the number of items which fail in a time interval of length  $t^*$ , is a random variable. If we have a replacement procedure, then the life test experiment becomes a Poisson process having a failure rate of  $\lambda = n/\theta$ . Known estimation procedures for the Poisson process can then be applied to find "best" estimates for  $\lambda$  and thus for  $\theta$ . In the non-replacement case

$$(3) \quad \hat{\theta} = \frac{\sum_{i=1}^k x_{i,n} + (n-k)t^*}{k}, \quad k = 0, 1, 2, \dots, n$$

is a maximum likelihood estimate for  $\theta$ . However it has no simple distribution. In the asymptotic sense (i.e., as  $n$  and  $k$  increase) we can say, however, that (3) is very nearly optimum.

An interesting class of estimation problems arises in the following way. Consider the decision procedures introduced at the beginning. In the course of reaching a decision data are collected. How can these data best be used to estimate  $\theta$ ? This will, of course, be recognized by many people as a problem which has aroused considerable interest of late. No very good answers can be given, except when the number of observations becomes large. We have found a number of results along these lines, which we think are quite useful. For example, we have shown how we can make best estimates when information comes from several experiments. Essential in this connection is a statistic which we call total life.

It is appropriate to mention certain results which may frequently be useful in practice, because of their great simplicity. Notice first that according to equation (2), no information is lost in the estimation

of the mean life  $\theta$  in the replacement case, if one uses  $x_{r,n}$  alone. This is equivalent to saying that  $x_{r,n}$  is sufficient for estimating  $\theta$  in the replacement case. In the non-replacement case  $x_{r,n}$  is not sufficient for estimating  $\theta$ . However, it is shown in A2 that very little information is lost (in the exponential case) if we retain only the value of  $x_{r,n}$  (the time of the  $r$ 'th failure) and forget about  $x_{1,n}, x_{2,n}, \dots, x_{r-1,n}$  (the times when the first  $(r-1)$  items failed). This result makes estimates of mean life and percentiles much easier to calculate. It also renders it unnecessary to maintain exact records of the time.

Some useful asymptotic results have also been obtained. By and large these show that excellent point and interval estimates of the mean life  $\theta$  can often be made, even if one does not know the exact times to failure. Thus, for example, consider a situation where 1000 tubes are each subjected to a 500 hour life test. It turns out that knowing  $k$ , the number of tubes which fail to survive 500 hours, is about as good as knowing all  $k$  failure times. Of course, one of our presuppositions is that the underlying p.d.f. is exponential.

### 3. Some other results on exponential life tests

In B4 we give a number of lemmas and theorems which are of interest in the general theory. These results furnish a basis for finding "best" estimates of  $\theta$  from knowledge about the times when the first  $r_1$  failures ( $r_1 \leq n_1, i = 1, 2, \dots, k$ ) occur in  $k$  different experiments. The underlying p.d.f.'s are now generalized so that they are two-parameter in nature, i.e., of the form  $\frac{1}{\theta} e^{-(x-A_1)/\theta}$ ,  $A_1 \geq 0, \theta > 0, i = 1, 2, \dots, k$ . The  $A_1$ 's may be the same or different, known or unknown.  $\theta$  is unknown and common to all  $k$  life distributions. The  $A_1$ 's can be interpreted as minimum life or as a sensitivity limit (in fatigue tests).

In C4, the work is extended to the Weibull distribution, where the s.d.f. of life is  $F(x;\theta) = 1 - e^{-(x/\theta)^b}$ ,  $x > 0$ ,  $\theta > 0$ . The exponential case corresponds to having  $b = 1$ . In some applications (for example, to ball bearing fatigue failures)  $b$  is known fairly well. This makes it possible to use exponential theory after certain minor modifications are made.

It was mentioned earlier that an important statistic in the exponential case is something we call "total life". Briefly, it is defined as follows. Let life testing commence at time  $t = t_0 = 0$  with  $n_0$  items. As the life test proceeds, there will be certain times  $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq \dots$ , when an abrupt change occurs. The  $t_i$ 's may be random variables or times fixed in advance. By an abrupt change we mean such occurrences as the failure of one of the items under test, the addition of new items to those still on test, or the removal of items from the test before they fail. If  $n_i$  is the number of items tested in the half-open interval  $[t_i, t_{i+1})$  then the observed total life  $T(t)$  in the interval  $[0, t]$  is defined as

$$(4) \quad T(t) = \sum_{i=0}^{k-1} n_i(t_{i+1} - t_i) + n_k(t - t_k).$$

In (4),  $k$  abrupt changes occur in  $[0, t]$ . In a manuscript called "Remarks on a Statistic Useful in Life Testing" we find the distribution of the statistic and some of its properties in special cases. This statistic makes it possible to treat estimation problems for replacement, non-replacement, truncated, suspended, and sequential life tests as special cases of a general theory. It will be seen in a later section that when people in inspection compute a life test rating for a sample of tubes, they are in reality finding the total life for a special situation.

#### 4. Some non-parametric life tests

In A6 and A7 we give some non-parametric two sample life tests. Taking advantage of the fact that observations occur in order, the decision procedure is terminated at time  $t = \max(x_r, y_r)$ , where  $r$  is preassigned and of course  $\leq n$ , the common sample size.

One of the non-parametric procedures, A6, makes essential use of the theory of exceedances, while the other procedure A7, is a truncated maximum deviation test of the Kolmogoroff, Smirnov, Massey, Z. W. Birnbaum type. In the course of this work tables which will be useful not only in life testing but also in flood and drought problems have been worked out. Exceedance tables are given for  $r = 1(1)n$  and  $n = 2(1)20(5)50$ . Tables of the distribution of the maximum deviation in truncated c.d.f.'s have been worked out for  $n = 1(1)10(5)40$  and  $r = 1(1) \left[ \frac{n}{2} \right]$ .

Of course, the big question in any non-parametric test is how good is the power? We have obtained experimental O.C. curves for the truncated maximum deviation and exceedance tests for the case where the underlying p.d.f.'s are normal with  $\sigma = 1$ . Two hundred pairs of samples of size 10 each were drawn from Wold's table of normal random deviates. Associating the letter X with one of the samples in each pair and Y with the other, we test X against Y, X against Y + 1, X against Y + 2, and X against Y + 3. Experimental O.C. curves were also found for the conventional run, rank sum, and maximum deviation tests (based on complete sample c.d.f.'s), using the same set of 200 pairs of samples. All tests were controlled in such a way that the probability of rejecting the null hypothesis when true was .05. We also found experimentally  $E_d(s)$ , the expected number of observations required to reach a decision, as a function of  $d = \mu_Y - \mu_X$  (the distance separating the means of the two normal distributions) for the exceedance and

truncated maximum deviation tests. The results indicate that while the exceedance and truncated maximum deviation tests have better power than the run test and worse power than the Wilcoxon rank test, there is a very substantial reduction in  $E_d(s)$ . This is due to the fact that we are taking advantage of the time ordered nature of the life test data. Another curious fact seems to be that for the class of alternatives studied,  $F(x)$  vs.  $F(x+d)$  ( $F$  normal), a truncated maximum deviation test has virtually as good power as does the standard maximum deviation test based on complete sample c.d.f.'s.

Of course we are well aware of the fact that any non-parametric test based only on the initial part of a distribution can be made to look bad against certain kinds of alternatives. However, there are certainly many cases in practice where the alternatives are sufficiently well-behaved or enough in known a priori to justify giving these procedures serious consideration. They are very simple to use and require so little computation.

#### V. Related work by other mathematical statisticians

At the time when this research was begun, the published literature dealing with the possibility of making use of order to reduce the time of experimentation or the number of observations or both was, as far as we can ascertain, limited to three papers (1,2,3). A resumé of these papers is given in B2.

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- (1) P. H. Jacobson, "The relative power of three statistics for small sample destructive tests", *Journal of the American Statistical Association*, 42, 575-584, 1947.
  - (2) J. F. Walsh, "Some estimates and tests based on the  $r$  smallest values in a sample", *Annals of Mathematical Statistics*, 21, 386-397, 1950.
  - (3) Max Halperin, "Maximum likelihood estimation in truncated samples", *Annals of Mathematical Statistics*, 2, 226-238, 1952.

In order to see the relevance of work done by others since the appearance of these three papers, we should mention the relationship between Poisson processes and exponential distributions. It is well known, for example, that the time interval between successive events in a Poisson process characterized by the rate of occurrence  $\lambda$ , is a random variable following a continuous probability law described by the exponential p.d.f.  $\lambda e^{-\lambda t}$ ,  $t > 0$ . It should be noted further that if one has  $n$  items on life test and replaces failures as soon as they occur by new items, then we are essentially dealing with a Poisson process having parameter  $\lambda = n/\theta$ , if the underlying p.d.f. of life is given by  $\frac{1}{\theta} e^{-x/\theta}$ ,  $x > 0$ ,  $\theta > 0$ . Thus it is clear that any results obtained on Poisson processes must be immediately applicable to exponential life test situations with replacement.

A. Birnbaum (4,5,6,7) has discussed this connection. He has found some sequential and non-sequential test for comparing Poisson populations which can be used to give two sample life tests.

Leo Goodman (8) has written an interesting paper on life testing. He takes up the problem of comparing the longevity of two or more type of equipment where it is not convenient to identify or keep records of

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- (4) A. Birnbaum, "Sequential tests and estimates for comparing Poisson populations" (Abstract No. 25, Annals of Mathematical Statistics 23, Dec. 1952).
  - (5) A. Birnbaum, "Some experimental designs for comparative life testing" (Abstract No. 14, Annals of Mathematical Statistics 24, March, 1953).
  - (6) A. Birnbaum, "Statistical methods for Poisson processes and exponential populations" (Mimeographed material distributed at Storrs, Connecticut, August, 1953).
  - (7) A. Birnbaum, "Some procedures for comparing Poisson processes and populations", to appear in Biometrika.
  - (8) Leo Goodman, "Methods of measuring useful life under operational conditions", Journal of the American Statistical Association, 48, 503-530, 1953.

individual items. He works under the assumption that all items are exposed to a constant risk, and this is equivalent to assuming an exponential p.d.f. He finds it convenient to treat the problem discretely (thus involving the negative binomial) while we have treated life testing from the continuous point of view. The introduction in Goodman's paper summarizes very well what the paper does. This introduction is now given verbatim.

"A comparison of the longevities of two or more types of equipment under operational conditions where it is not convenient to identify or keep records of individual items can be made by adopting a certain replacement policy and observing its effect on the composition of the population. When only two types are being compared, for example, the policy might be that when an item fails it will be replaced by one of the opposite type. Then the composition of the population at any time (that is, the proportions of the different types among all the items in use) will depend upon the original composition of the population, the time elapsed, and the longevities of the different types. Since the original composition and the elapsed time are known, by determining the new composition of the population we can obtain information concerning the longevities of the different types of equipment."

An interesting and intrinsic feature of sequential life test procedures is that decisions can be made continuously in time. This raises some new and important theoretical questions. We posed the problem as it arises in life testing in C1. The solution is outlined in B1 and given in detail in A8. One of the main results is a fundamental identity relating  $E_0(t)$ , the expected waiting time to reach a decision and  $E_0(r)$ , the expected number of items failed. This turns out to be a consequence of a general



theorem by Doob (9) on continuous parameter martingales. Our proof in A3 was obtained independently of Doob's work and without using the heavy technical machinery required to handle the general problem treated by Doob.

Recently Dvoretzky, Kiefer, and Wolfowitz (10,11) have treated sequential procedures in stochastic processes with a continuous time parameter. When the stochastic process treated is Poisson, the results can be used in life testing. We might add that a great deal of numerical work done here has shown that the approximations that we give in A8 for  $E_G(t)$  and  $E_G(r)$  are good enough for all practical purposes. The exact formulae (7) require a great deal of time and labor.

Material relevant to some of the estimation problems in life testing can be found in work on the Poisson distribution by a number of British statisticians (12,13,14,15,16).

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- (9) J. L. Doob, "Stochastic processes", John Wiley and Sons, 1953.
  - (10) A. Dvoretzky, J. Kiefer, and J. Wolfowitz, "Sequential decision problems for processes with continuous time parameter. Testing hypotheses", *Annals of Mathematical Statistics*, 24, 254-264, 1953.
  - (11) A. Dvoretzky, J. Kiefer, and J. Wolfowitz, "Sequential decision problems for processes with continuous time parameter. Problems of estimation", *Annals of Mathematical Statistics* 24, 403-415, 1953.
  - (12) D. G. Kendall, "Stochastic processes and population growth", *Journal of the Royal Statistical Society, Ser. B.*, 11, 230-264, 1949.
  - (13) P. A. P. Moran, "Estimation methods for evolutive processes", *Journal of the Royal Statistical Society, Ser. B.*, 13, 141-146, 1951.
  - (14) F. J. Anscombe, "Sequential Estimation", *Journal of the Royal Statistical Society, Ser. B.*, 15, 1-29, 1953.
  - (15) M. S. Bartlett, "Approximate confidence intervals", *Biometrika*, 40, 12-19, 1953.
  - (16) M. S. Bartlett, "On the statistical estimation of mean life time, *Philosophical Magazine* 44, 1953.

Our work on exceedances extends earlier work of Gumbel and von Schelling (17) and has relevance to the work of Mosteller and Tukey (18,19) on slippage.

Our studies on the power of non-parametric life tests have connections on the theoretical side with recent work by Lehmann (20) and on the empirical side with sampling studies on the comparative power of various non-parametric tests by Teichroew and Dixon (21,22).

#### VI. Some remarks on the applications of this work

In B1 we devote some space to the exponential distribution and its role in the life testing of electron tubes. For convenience we bring together several references (23,24,25,26,27,28) dealing with this question. Reference

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- (17) E. J. Gumbel and H. von Schelling, "The distribution of the number of exceedances", *Annals of Mathematical Statistics*, 21, 247-262, 1950.
  - (18) Frederick Mosteller, "A k-sample slippage test for an extreme population", *Annals of Mathematical Statistics*, 19, 58-65, 1948.
  - (19) F. Mosteller and J. W. Tukey, "Significance levels for a k-sample slippage test", *Annals of Mathematical Statistics*, 21, 120-123, 1950.
  - (20) E. L. Lehmann, "The power of rank tests", *Annals of Mathematical Statistics*, 24, 23-42, 1953.
  - (21) D. Teichroew, "Empirical sampling distributions II", N.B.S. report 1621, April, 1952.
  - (22) W. J. Dixon and D. Teichroew, "Some sampling results on the power of non-parametric tests against normal alternatives", paper presented at Institute of Mathematical Statistics, December 1953.
  - (23) D. J. Davis, "An analysis of some failure data", *Journal of the American Statistical Association*, 47, 113-150, 1952.
  - (24) C. R. M. Tuttle and A. R. Frank, "Inventory control methods applied to electronic tubes", a paper included in *Logistics Papers*, Issue 8, November 16, 1951 - February 15, 1952, Appendix I. Issued by the George Washington University Logistics Project.
  - (25) E. S. Rich, "Experience with receiving type vacuum tubes on the Whirlwind Computer Project", Report R-194, Project Whirlwind, Servomechanisms Laboratory, Massachusetts Institute of Technology, February, 1951.

23 is quite general and includes examples from a variety of fields while the rest of the references deal primarily with electron tube life.

While the evidence contained in the papers just cited strongly suggests that the exponential distribution of life is a reasonable assumption to make in the case of tube failures, it is probably only a first approximation to reality. For example in reference 28 the exponential p.d.f. of life was obtained after eliminating a large number of early failures due to all kinds of mechanical defects, shorts, open circuits, etc. The same point has been emphasized in recent papers by McElwee and others (29,30).

During the past two years the subject of Life Testing has been a topic of discussion on three different occasions at national meetings. The first of these took place in July, 1952, at the Summer Statistics Conference sponsored by the University of North Carolina and held at Blue Ridge, North Carolina. Participating in the discussion were J. R. Steen and R. D. Wilde

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- (26) N. W. Lewis, "Note on the exponential distribution in statistics", Post Office Electrical Engineers Journal 41, 10-12, 1948.
  - (27) D. K. Gannett, "Determination of the average life of vacuum tubes", Bell Telephone Laboratory Record, 373-381, 1940.
  - (28) J. E. M. Lagasse and W. H. Dean, "Tubes at Work", Electronics, 56-62, 1941.
  - (29) M. A. Acheson and E. M. McElwee, "Concerning the reliability of electron tubes", The Sylvania Technologist, 4, April, 1951.
  - (30) A. J. Heitner and A. L. Wilson, "Analysis of Kew Gardens premium tube life", Sylvania Electric Products, Inc. 83-30 Kew Gardens Road, Kew Gardens 15, Long Island, New York, May, 1951.

of Sylvania Electric Products, Inc., J. A. Davies of General Electric Company, and J. R. Ransom of DuPont. The present writer was unable to attend the conference in person, but does have mimeographed copies of the papers presented by Steen, Wilde, and Davies (31,32,33,34). The following remarks are pertinent:

- (1) All of the speakers stressed the existence of essentially two kinds of failures, one of which occurs early in life, the other of which is associated with a gradual and prolonged wear-out process. It was generally agreed that if the first kind of failure would be eliminated, then the assumptions of a constant failure rate (i.e., exponential life) was reasonable.
- (2) It would seem that the results obtained under this research contract are laying the foundations for a rational inspection test procedure. Steen stated in his talk that to be acceptable a life test procedure must yield the required information about the tubes as economically as possible, i.e., should provide a basis for action which minimizes the over-all test cost. This means getting the most information in the shortest possible time and with the smallest number of failures. This is precisely the objective of the research project.

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- (31) J. R. Steen, "Life testing of electronic tubes".
  - (32) R. D. Wilde, "Nature of rate of failure curves".
  - (33) R. D. Wilde, "An example of a statistically designed experiment in a radio tube manufacturing process".
  - (34) J. A. Davies, "Life test acceptance sampling methods".

Items 30, 31, 32, 33 were all presented at the North Carolina Conference.

- (3) Wilde found the results of A1 very useful in estimating tube life.
- (4) Davies' concern was with a variety of sampling plans such as the JAN-1A, JTC-11, and the so-called G.E. Plan. One of the essential features of all of these plans is that the decision to accept or reject a lot is based on a quantity known as a life test rating. This is widely used in the electrical industry and is described in various places (29,35,36). Briefly, a life test rating is computed as follows. Let  $n$  tubes be run for a maximum of  $T$  hours. Suppose  $r$  tubes fail at times  $x_1, x_2, \dots, x_r$  before  $T$ . Then the total observed life is defined as 
$$U = \sum_{i=1}^r x_i + (n-r)T,$$
 and the life test rating in percent is simply  $100 U/nT$ . The rule of action is to accept a lot if the life test rating based on the sample exceeds some value (e.g., 90%). Otherwise the lot is rejected.

We should like to remark that the quantity  $U$  and others like it arise in a natural way when one deals with an underlying life distribution which is exponential. As mentioned at the end of section IV of this report, theoretical work on the statistic which we call "total life" is very relevant here. Both Davies and Steen stress the need for developing analytical methods for finding the O.C. curves for procedures such as the G.E. plan. No such methods exist for the G.E. plan and apparently it has been necessary for Davies to fall back on empirical sampling procedures to gain such information. It turns out that the work we have done on truncated test procedures (A4, B5) is almost made to order in this connection. The truncated test procedures, while not identical with the G.E. procedure, do have the same

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- (35) E. H. McElwee, "Statistical evaluation of life expectancy of vacuum tubes designed for long life operation", Proceedings of the Institute of Radio Engineers, February, 1951.
- (36) J. M. Juran, "Quality Control Handbook", Chapter 13, McGraw-Hill, 1951.

objective of putting a fixed upper limit on the time of experimentation. In addition, our truncated tests possess the desirable feature that all pertinent properties including the O.C. curve can be worked out analytically in any case of practical interest. It should be noted that the sampling experiments of Davies were performed under the assumptions of a constant rate of failure. This means that he is assuming implicitly that the underlying life distribution is exponential.

At the Seventh Annual Convention of the American Society for Quality Control held in Philadelphia in May, 1953, there were two sessions dealing with life testing. One of these was on the topic "Reliability of Electronic Equipment". Papers were given by Herd (37) and Guild (38). Herd is in charge of statistical work at Aeronautical Radio, Inc., which has the responsibility of investigating the reliability of military electronic equipment under field conditions. In this work, Herd has found the results of our research to be very useful.

The second session at the convention dealing with life testing was on the topic "Life Testing Electrical and Electronic Equipment". Papers were given by Epstein (B1) and by Davies (39). In B1 the research work done on the statistics of life testing under this project was outlined. Paper 39 is essentially the same as Paper 34, i.e., is concerned with various sampling plans to judge the acceptability of a lot on the basis of what happens to a sample on life test.

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- (37) G. R. Herd, "Application of statistical methods in evaluating performance of electronic equipment", Proceedings of the Seventh Annual Convention of the American Society for Quality Control, pp. 309-314, 1953.
  - (38) R. D. Guild, "Statistical appraisal of vacuum tube reliability", Proceedings of the Seventh Annual Convention of the American Society for Quality Control, pp. 225-230, 1953.
  - (39) J. A. Davies, "How reliable is your life test procedure?" Proceedings of the Seventh Annual Convention of the American Society for Quality Control, pp. 255-266, 1953.

On August 26-28, 1953, there was a seminar on "Performance and Reliability of Complex Assemblies" held at the University of Connecticut, Storrs, Connecticut. About forty people from all parts of the country attended. Government, industry, and universities had representatives there. Much of the discussion concerned itself with problems of life and fatigue of components, sub-assemblies, and assemblies. Paper B1 was presented at this conference.

In connection with problems of reliability we should like to mention an interesting paper by E. W. Pike and F. C. Lewis (40) given before the Institute of Radio Engineers in Dayton in May, 1953.

The research on life testing is giving new statistical tools for people in the field of operations research. This was discussed in a recent paper by R. L. Ackoff (41). The following pertinent statement is taken from the first page of his paper:

"Many operations research problems involve questions concerning the maintenance, replacement, or critical strengths of equipment or of personnel. For example, in the military, how often should an aeroplane motor be replaced, or torn down, or rebuilt? How long should a pilot or infantryman remain in combat? How much pressure can a certain thickness of armor plate withstand? In industry and business there are similar questions. How often should light bulbs, or electronic tubes, or transistors, or tires be replaced? When should an automobile motor be torn down and rebuilt? How long should a worker be kept on a repetitive process? Such questions as these deal with what is called the life-span or fatigue point of the items or personnel involved."

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(40) E. W. Pike and F. C. Lewis, "Testing Components for High-Reliability Systems", mimeographed 1953.

(41) R. L. Ackoff, "Some new statistical techniques applicable to operations research", Journal of the Operations Research Society of America 1, 10-17, 1952.



Some applications of this work are being made in areas other than electron tubes. For example, Besse Day and her associates at the U. S. Naval Experiment Station, Annapolis, Md., are interested in ball-bearing life. They are trying out some of the exponential techniques developed here as subsequently modified by Johnson of General Motors in C4 to apply to the Weibull distribution. This distribution seems to be quite useful in fitting data arising in a variety of applications (42,43). The close connection between statistical theories of strength and the theory of extreme values was developed by the present writer a few years ago (44) and this explains in part why the Weibull distribution plays an important part in applications. A recent paper of Freudenthal and Gumbel (45) is relevant here. Dr. Day has found the Weibull assumption to be reasonably satisfactory.

Some interesting work on battery life is being done by Dr. William Buckland who is head of the Statistical Section in the Research and Development Section of the London Transport Executive. He is carrying out extensive studies on the distribution of life of batteries used in diesel buses. The life distributions are expected to be Type III or Weibull rather than exponential.

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- (42) W. Weibull, "A statistical theory of strength of materials", Vetenskaps Akad. Hand. No. 151, 1939, "The phenomenon of rupture in solids", *ibid.*, No. 153, 1939.
  - (43) W. Weibull, "A statistical distribution of wide applicability", *Journal of the Applied Mechanics*, 239-297, September, 1951.
  - (44) B. Epstein, "Statistical aspects of fracture problems", *Journal of Applied Physics*, 19, 140-147, 1948.
  - (45) A. M. Freudenthal and E. J. Gumbel, "On the statistical interpretation of fatigue tests", *Proceedings of the Royal Society of London, Series A*, 216, 309-332, 1953.



J. A. Greenwood of the U. S. Navy Bureau of Aeronautics has done some interesting work on a semi-empirical procedure for estimating the mean life of turbine blocks. Life in hours seems to be normally distributed in this application.

The last three applications arose from personal contacts and correspondence. It might be mentioned that a steadily increasing amount of correspondence indicates that there is considerable interest in the practical applications of the work. Scores of governmental and industrial organizations have written to inquire about various aspects of the research. There is no doubt in my own mind that the work carried out under this project will in time find its application in many places, including the biological as well as physical sciences.

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